Exploring generalization with visual patterns: tasks developed with pre-algebra students[1]

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RESUMO

Este artigo refere-se a um estudo desenvolvido com 54 alunos do 6.º ano de escolaridade. O principal objectivo passava por analisar o seu desempenho na resolução de tarefas que envolviam a generalização de padrões visuais. De forma a compreender este problema centramo-nos nas seguintes temáticas: tipo de estratégias de generalização utilizadas; dificuldades que emergiram do seu trabalho; e o papel da visualização no seu raciocínio. Neste artigo são apresentados alguns resultados relativos à implementação de duas tarefas.

Palavras-chave: padrões, generalização, visualização.

ABSTRACT

This paper refers to a study developed with fifty-four 6th grade students. The main goal was to analyse their performance when solving tasks involving the generalization of visual/figural patterns. In order to better understand this problem we focussed on the following features: type of generalization strategies used; difficulties that emerged from students' work; and the role played by visualization on their reasoning. On this paper we present some results related to the implementation of two tasks.

Key words: patterns, generalization, visualization.

INTRODUCTION

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The 80's represent a landmark for school mathematics. At that time, profound curricular changes were made and problem solving became an integral part of all mathematics learning (NCTM, 2000). This idea is still present in the recent curricular guidelines of several countries, where the ability to solve problems is mentioned as one of the main goals of learning mathematics. In spite of the relevance given to this theme, some international studies (SIAEP, TIMSS, PISA) have shown that Portuguese students perform badly when solving problems (Ramalho, 1994; Amaro, Cardoso & Reis, 1994; OCDE, 2004). Pattern exploration tasks may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data in a systematic way, conjecturing and generalizing. For instance, the *Principles and Standards for School Mathematics* (NCTM, 2000) acknowledges the importance of working with numeric, geometric and pictorial patterns. This document states that instructional mathematics programs should enable students, from pre-kindergarten to grade 12, to engage in activities involving understanding patterns, relations and functions. Work with patterns may also be helpful in building a more positive and meaningful image of mathematics and contribute to the development of several skills, in particular related to problem solving and algebraic thinking (Vale et al, 2006). On the other hand, Geometry is considered a source of interesting problems that can help students develop abilities such as visualization, reasoning and argumentation. Visualization, in particular, is an important mathematical ability but, according to some studies, its role hasn't always been emphasized in students' mathematical experiences (Healy & Hoyles, 1996; Presmeg, 2006). Although the usefulness of visualization is being recognized by many mathematics educators, in Portuguese classrooms teachers privilege numeric aspects over geometric ones (Vale & Pimentel, 2005). Considering it all, we think that more research is still necessary concerning the role images play in the understanding of mathematical concepts and particularly in problem solving.

This study intends to analyse the performance of 6th grade students (11-12 years old) when solving problems involving visual/figural patterns. The tasks used in the study require pattern generalization and students of this age have not yet had formal algebra instruction, thus the importance of analysing the nature of their approaches. This study attempts to address the following research questions:

1) Which difficulties do 6th grade students present when solving pattern exploration tasks?

2) How can we characterize students' generalization strategies?

3) What's the role played by visualization on students' reasoning?

THEORETICAL FRAMEWORK

Patterns in the teaching and learning of mathematics

Many mathematicians share an enthusiastic view about the role of patterns in mathematics, some even consider mathematics as being the science of patterns (Steen, 1990). This perspective highlights the presence of patterns in all areas of mathematics, considering it a transversal and unifying theme. In particular, the search for patterns is seen by some investigators as a way of approaching Algebra since it is a fundamental step for establishing generalization, which is the essence of mathematics (Mason, Johnston-Wilder & Graham, 2005; Orton & Orton, 1999; Zazkis & Liljedahl, 2002). The mathematics curricula of many countries contemplate significant components like: searching for patterns in different contexts; using and understanding symbols and variables that represent patterns; and generalizing. Portuguese curriculum mentions the importance of developing abilities like searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing and generalizing (DEB, 2001; ME, 2007). These abilities are directly related to algebraic thinking and support the development of mathematical reasoning as well as the connection between mathematical ideas (NCTM, 2000).

Patterns and generalization

There has been considerable research concerning students' generalization strategies, from pre-kindergarten to secondary school. We proceeded to an adaptation of some frameworks proposed by different investigators (Lannin, 2003; Lannin, Barker & Townsend 2006; Orton & Orton, 1999; Rivera & Becker, 2005; Stacey, 1989; Swafford & Langrall, 2000) and came up with the following categorization:

Strategy		**Description**
Counting (C)		Drawing a figure and counting the desired elements.
Whole-object	No adjustment (W_1)	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment (W_2)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment (W_3)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Difference	Recursive (D_1)	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment (D_2)	Using the common difference as a multiplying factor without proceeding to a final adjustment.
	Rate - adjustment (D_3)	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
Explicit (E)		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
Guess and check (GC)		Guessing a rule by trying multiple input values to check its' validity.

Table 1. Generalization Strategies Framework

Analyzing previous research and drawing on personal experiences, we recognized that these strategies often emerge through different types of reasoning and we think that it's fundamental that students understand the potential and limitations of each approach. Depending on the type of task, some of these strategies may lead students to difficulties or even incorrect answers. This fact is reported on a variety of studies:

- an incorrect application of the direct proportion method, mainly when attempting to generalize linear patterns (Lannin, Barker & Townsend 2006; Rivera & Becker, 2005; Sasman, Olivier & Linchevski, 1999; Stacey, 1989). Students often operate exclusively on numeric contexts, manipulating variables without recognizing their meaning;
- the fixation on a recursive approach that, although being useful in solving near generalization tasks, doesn't contribute to the understanding of the structure of a pattern (Orton & Orton, 1999). Frequently, when dealing with far generalization, students apply a recursive rule based on multiples of the common difference, neglecting to correct the result through the adjustment to the context of the problem (Lannin, Barker & Townsend 2006; Sasman, Olivier & Linchevski, 1999);
- guess and check is a well know strategy for solving problems, often encouraged in our classrooms. This approach is considered a good problem solving numerical tool, but it can lead students to incorrect conclusions if all conditions aren't considered (Rivera & Becker, 2005);
- functional reasoning proves to be very complex for many students, especially those from elementary levels. Some of these difficulties are due to: using improper language to describe relations; the extensive use of recursive reasoning and the guess and check strategy; and the inability to visualize or explore patterns spatially (Warren, 2008).

Thinking preferences and generalization

Patterning activities can be developed in a variety of contexts (numeric, geometric, concrete and figural) and through the application of different approaches. Gardner (1993) claims that some individuals recognize regularities spatially or visually, while others notice them logically or analytically. In fact, it is common, in mathematical activities, that different individuals process information in different ways. Many students favour analytic methods while others have a tendency to reason visually.

The relation between the use of visual abilities and students' mathematical performance constitutes an interesting area for research. Many investigators stress the importance of the role visualization plays in problem solving (Presmeg, 2006; Shama & Dreyfus, 1994), while others claim that visualization should only be used as a complement to analytic reasoning (Goldenberg, 1996; Tall, 1991). In spite of some controversy, these visions reflect the importance of using and developing visual abilities in mathematics but teachers tend to present visual reasoning only as a possible strategy for problem solving in an initial stage or, when necessary, as a complement to analytic methods (Presmeg, 1986). Several studies point to the potential of visual approaches for supporting problem solving and mathematical learning. The reality of our classrooms, however, tells us that students display frequently reluctance to exploit visual support systems (Dreyfus, 1991) and tend not to make links between visual and analytical thought (Presmeg, 1986).

Some studies report the impact of the nature of the approaches used by students to generalize, referring, in particular, to the relevance of visual abilities. García-Cruz & Martinón (1997) developed a study aiming to analyse the processes of generalization developed by secondary school students. They classified generalization strategies according to their nature: *visual*, *numeric* and *mixed*. If the drawing played an essential role in finding the pattern it was considered a *visual* strategy, on the other hand, if the basis for finding the pattern was the numeric sequence then the strategy was considered *numeric*. Students who used *mixed* strategies acted mainly on the numeric sequence and used the drawing as a means to verify the validity of the solution. Results of this research have shown that the drawing played a double role in the process of abstracting and generalizing, in both cases fundamental. It represented the setting for students who used visual strategies in order to achieve generalization and acted as a means to check the validity of the reasoning for students who favoured numeric strategies. In a more recent study, Becker and Rivera (2005) described 9th grade students work after they were asked to perform generalizations on a task involving linear patterns. They tried to analyse successful strategies students used to develop an explicit generalization and to understand their use of visual and numerical cues. The researchers found that students' strategies appeared to be predominantly numeric and identified three types of generalization: *numerical*, *figural* and *pragmatic*. Students using numerical generalization employed trial and error with little sense of what the coefficients in the linear pattern represented. Those who used *figural* generalization focused on relations between numbers in the sequence and were capable of seeing variables within the context of a functional relationship. Students who used *pragmatic* generalization employed both numerical and figural strategies, seeing sequences of numbers as consisting of both properties and relationships.

METHOD

Fifty four sixth-grade students (11-12 years old), from three different schools in the North of Portugal, participated in this study over the course of a school year. The study was divided in three stages: the first corresponded to the administration of a test focusing on pattern exploration and generalization problems; second stage, which went on for six months, involved all students in each classroom solving patterning tasks, in pairs; and, on the third, students repeated the test in order for us to examine changes in the results. These students were described by their teachers as being of average ability and had no prior experience with this kind of tasks. Over the school year all students involved in the study solved seven tasks and two pairs from each school were selected for clinical interviews. Students' activity when solving the tasks was videotaped and transcribed for further analysis.

The tasks applied along the study required near and far generalization and featured increasing and decreasing linear patterns as well as non linear ones. In the selection process we tried to privilege tasks whose structure could lead to the use of multiple strategies, allowing students to find patterns in either numeric or visual contexts.

In this paper we report some results from the application of two of the tasks.

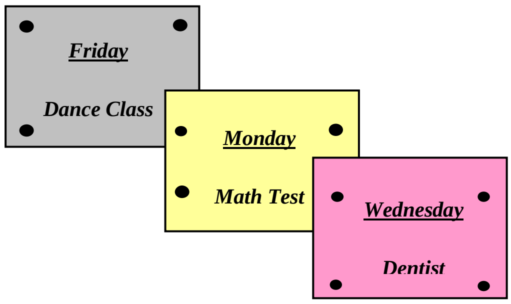
PRELIMINARY RESULTS

The *Pins and Cards* task

One of the selected tasks was called *Pins and Cards* (figure 1). This was the first task solved by the students in this study and represents an increasing linear pattern, presented in a figural form. The first question called for near generalization, by finding the 6th term of the sequence, while the last two questions required far generalization.

Pins and Cards

Joana hangs cards on a board in her room in order to remember her appointments. She uses pins to support the cards as shown in the image.



If she continues to hang cards in her board this way:

1. *How many pins will she need to hang 6 cards?*
2. *What if she was to hang 35 cards, how many pins would she need?*
3. *Supposing that Joana bought a box with 600 pins, how many cards can she hang in her board?*

Figure 1. *Pins and Cards* task

The analysis of students' work allowed us to identify a diversity of generalization strategies, as well as some difficulties approaching particular questions.

We start to present in table 2 the number of pairs of students that used a given strategy, based on the categories described on the *Generalization Strategies Framework* (table 1). In some cases we couldn't categorize students' answers so those cases appear in the last column of the table, not categorized (*NC*). This table allowed us to analyze not only the approach used to solve each of the questions of the task, but also compare it with the level of generalization involved (near or far).

	C	W_1	W_2	W_3	W	D_1	D_2	D_3	D	E	GC	NC
1.	16	8	-	1	9	1	1	-	2	-	-	-
2.	-	3	2	1	6	3	1	-	4	12	-	5
3.	-	2	1	-	3	-	4	3	7	9	-	8

Table 2. Summary of the strategies used by the students

Near generalization strategies

The first question of this task required near generalization. This type of questions can easily be solved by making a drawing of the requested term of the sequence and counting its elements, using the *counting* strategy. Figure 2 represents an example of this approach, to find the number of pins, presented by a pair of students.

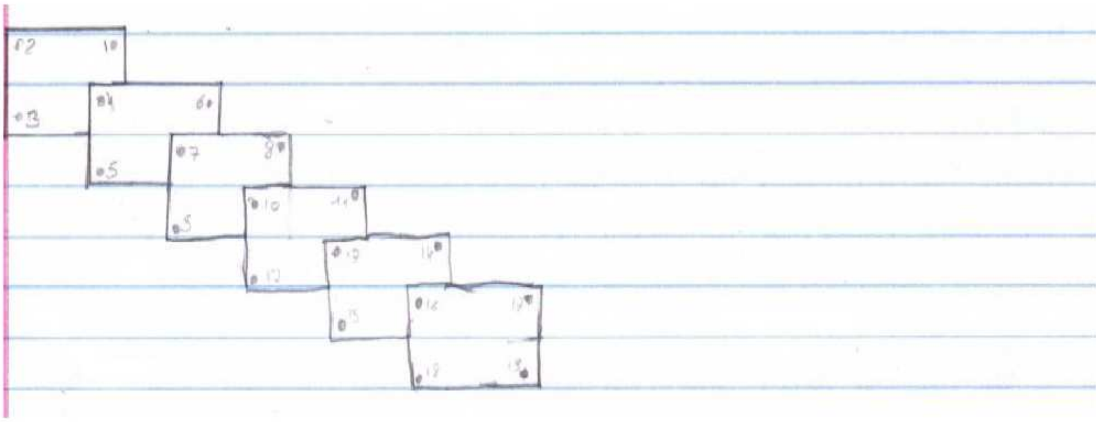


Figure 2. Example of the *counting* strategy.

As we can see from table 2 *counting* over a drawing was the predominant strategy in near generalization, always leading to a correct answer.

The *whole-object* strategy also emerged from the work of some of the pairs. This approach is associated to direct proportion situations and this particular problem does not fit this model. Nevertheless, eight pairs of students used proportional reasoning, duplicating the number of pins associated to the three cards. For this strategy to be adequate, students had to make a final adjustment based on the context. Only one of the pairs felt the need to adjust the result obtained in the duplication of the number of pins of the three cards (figure 3).

3+3=6 cards

10+10=20 pins *These two groups share 1 pin.*

20-1=19 pins

Figure 3. Example of the W_3 strategy.

According to the existing literature (e. g. Orton & Orton, 1999; Stacey 1989) this type of tasks can promote the use of *recursive* thinking, especially when near generalization is involved. Curiously only one pair of students extended the sequence using the common difference to solve this question. We registered one other case, in which the *difference* strategy was employed but in an incorrect way. To obtain the number of pins necessary to hang 6 cards, these students used a multiple of the common difference without adjusting the result, as happened in other cases with the *whole-object* strategy.

The *explicit* and *guess and check* strategies were not applied to solve this question.

Far generalization strategies

Although both questions 2 and 3 require far generalization, the third question of the task had a different structure, involving reverse thinking. When approaching far generalization students revealed more difficulties and that can be seen by the increasing number of not categorized answers, that represent imperceptible reasoning or no answer at all (table 2).

We noticed in table 2 that students dropped the *counting* strategy when solving these two questions. Some pairs did start by using it but gave up along the way, claiming that “there were too many cards”. Instead, the application of *explicit* strategies prevailed. Those who relied on this approach, using the context to identify an immediate relationship between the two variables, presented a high level of efficacy. Some students “saw” that each card needed three pins and the last one would need four, deducing that the rule was *3(n-1)+4*, *n* being the number of cards. Other pairs “saw” the pattern differently considering that each card had three pins adding one more pin at the end. Here the rule was *3n+1*. In fact, research on pattern and generalization shows that individuals might see the same pattern differently (Rivera & Becker, 2007), originating equivalent expressions.

The *whole-object* strategy continued to appear as in the previous question (1), but this time a new approach emerged. Some students considered multiples of known terms of the sequence and adjusted the result based only on numeric properties (figure 4).

6 *cards*	-	20 *pins*	
12 *cards*	-	40 *pins*	
18 *cards*	-	60 *pins*	
24 *cards*	-	80 *pins*	
30 *cards*	-	100 *pins*	120-4=116 *pins*
36 *cards*	-	120 *pins*	36-1=35 *cards*

Figure 4. Example of the W_2 strategy.

The example given on figure 4 shows that students rely on proportional reasoning to determine the number of pins and, when adjusting the result they don’t consider the context of the problem, only numeric properties, obtaining an incorrect answer.

Comparing the first question with the last two, we can see that the use of the *difference* strategy increases. Some students gave up on *counting*, as the order of the term became far, and started basing their reasoning on the common difference between terms. In the third question of the task, we noticed that three pairs of students applied a strategy that hasn’t been used before.

3×200=600 pins

But we need one more pin for the last card, so she only can hang 199 cards.

Figure 5. Example of the D_3 strategy

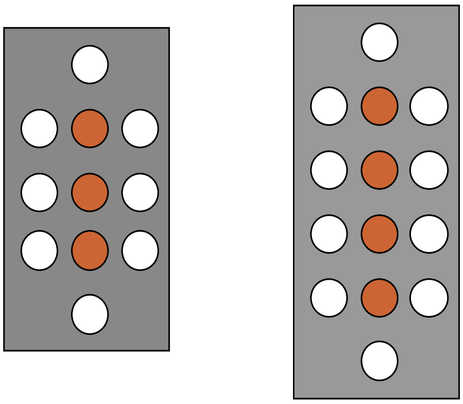
The difference between consecutive terms is of three pins, so, in this case, students used that fact to approach the number of pins available. Knowing the structure of the pattern, they were able to criticize the result, adjusting it.

The *Sole Mio Pizzeria task*

This task was solved four months later. The problem is similar to the one presented on the previous task, exhibiting an increasing linear pattern and contemplating near and far generalization questions.

Sole Mio Pizzeria

The image shows two tables of the Sole Mio Pizzeria, one with 8 people and 3 pizzas and the other with 10 people and 4 pizzas.



1. *How many people would sit in a table with 10 pizzas?*
2. *What if the table had 31 pizzas? How many people would be sited?*
3. *John decided to celebrate his birthday in this Pizzeria. Knowing that he invited 57 guests, how many pizzas would he have to order?*

Figure 6. *Sole Mio Pizzeria* task

In order to compare the strategies, selected by students in this task, with the strategies used in the *Pins and Cards* task, we organized the categories in the following table:

	C	W_1	W_2	W_3	W	D_1	D_2	D_3	D	E	GC	NC
1.	21	-	-	-	-	4	-	-	4	2	-	-
2.	1	-	-	-	-	3	-	1	4	22	-	-
3.	-	-	-	-	-	2	3	-	5	14	5	3

Table 3. Summary of the strategies used by the students

One of the most obvious facts is the lack of preference for the *whole-object* strategy. Being a linear pattern, the use of proportional reasoning is not adequate, unless an adjustment based on the context is made. We believe that, in this case, the adjustment was more complex than in the previous problem which could justify the absence of this approach.

Near generalization strategies

Counting is once again the privileged strategy in near generalization. It is applied by the majority of the students and this preference has increased compared to the previous task.

Other strategies emerged but only a minority of students used them. Four pairs used recursive reasoning to extend the sequence to the 10^{th} term and two pairs applied an explicit reasoning. In the first task, explicit strategies only appeared when students were dealing with far generalization so it's surprising that they used it at this stage, showing that they immediately discovered the structure of the pattern.

Far generalization strategies

As in the *Pins and Cards* task, when dealing with far generalization, students do not recognize the usefulness of *counting* and that's why it has no expression on table 3, as we progress to far generalization. On the other hand, *explicit* reasoning prevails being implemented by even more students in a successful way. Curiously all of them described the pattern as *2n+2*, *n* being the number of pizzas. They frequently referred that "in front of each pizza are two people and one more at each end of the table".

Some students chose a safe path going with a recursive approach, through the extension of the sequence using the common difference. Similarly to what happened in the previous task, there were three pairs that considered multiples of the common difference but neglected to adjust the result, showing that their work was merely based on number relations.

We also noticed the use of a new strategy, guess and check, that was only applied in far generalization when reverse thinking was involved.

24+24+2=50 *25+25+2=52* *26+26+2=54* *27+27+2=56* *28+28+2=58*	He would have to order 28 pizzas.

This example shows that the students identified the relation between the two variables in the case that the independent variable was the number of pizzas and the independent variable was the number of people. Based on that rule they tried some numbers until they achieved the result wanted.

Difficulties emerging from students work

When solving the first task some students struggled with cognitive difficulties that led to incorrect answers. Some pairs made false assumptions about the use of direct proportion. In these cases attention tended to be focused only on numeric attributes with no appreciation of the structure of the sequence. This happened with strategies W_1 and W_2, where the only concern was to satisfy numeric relations. The use of strategies based on recursive reasoning wasn't always made correctly, especially when far generalization questions were involved. The recursive approach through the use of D_2 lacked a final adjustment based on the context of the problem, because students only considered a multiple of the common difference, forgetting to add the last four pins or the last pin, depending on the interpretation. Also, when they used explicit strategies, the model wasn't always correctly applied. In some cases, students added pins and cards in the end. We are convinced that these errors are linked to the extensive experience of students in manipulating numbers without meaning, making no sense of what the coefficients in the linear pattern represent.

Analyzing tables 4 and 5 we verify that the level of efficacy presented by students increases on the second task. There is more awareness on the selection of the proper strategies to use in each case, for example, the inadequate use of direct proportion is no longer observed. In spite of these differences, we still notice that students experience difficulties when reverse thinking is involved, being also clear that this type of questions provokes a shift on the type of approaches used by them.

Pins and Cards	**C**	**W**	**D**	**E**	**GC**
Near generalization	16	9	2	-	-
% of efficacy	100%	11%	50%	-	-
Far generalization	-	6	4	12	-
% of efficacy	-	0%	75%	100%	-
Far generalization (reverse thinking)	-	3	7	9	-
% of efficacy	-	0%	43%	89%	-

Table 4. Level of efficacy per strategy

The Sole Mio Pizzeria	**C**	**W**	**D**	**E**	**GC**
Near generalization	21	-	4	2	-
% of efficacy	100%	-	100%	100%	-
Far generalization	1	-	4	22	-
% of efficacy	100%	-	100%	100%	-
Far generalization (reverse thinking)	-	-	5	14	5
% of efficacy	-	-	40%	100%	100%

Table 5. Level of efficacy per strategy

The role of visualization in students' reasoning

According to Presmeg (1986) a strategy is considered visual if the image/drawing plays a central role in obtaining the answer, either directly or as a starting point for finding the rule. In this sense we believe that the following strategies are included in this group: *counting*, *whole-object with visual adjustment*, *difference with rate-adjustment* and *explicit*.

Counting was always a successful strategy but only useful in solving near generalization questions. Drawing a picture of the object required and counting all the elements is an action used in near generalization questions and does not lead to a generalized strategy.

Strategy W_3 was only used by one pair of students, when solving the first task. They've only applied correctly in near generalization. We think that this kind of reasoning involves a higher level of abstraction in visualization, difficult to attain.

In spite of not being one of the most frequent strategies, students who used D_3 always reached the correct answer. This fact reflects once more the relevance of understanding the context surrounding the problem, making the relation between variables clearer.

Finally, the application of an *explicit* strategy lead to a high level of efficacy. Students based their work on the structure of the sequence, making reference to the relation between the variables reported on the problem. We registered only a few cases that, along the way, disconnected from the context and mixed different variables.

DISCUSSION

In this research, the main purpose of using pattern exploration tasks was setting an environment to analyse difficulties presented by students, strategies emerging from their work and the impact of using visual strategies in generalization.

As for the research questions outlined earlier in this paper, we can now make some observations: (a) students achieved better results in near generalization questions than on far generalization questions and, even with some experience with patterning activities, reverse thinking was still complex for many of them; (b); a variety of strategies were identified in the work developed by students, although some were more frequent than others, like *counting* (mostly on near generalization) and *explicit* (more frequent on far generalization); (c) some of the pairs worked exclusively on number contexts using inadequate strategies like the application of direct proportion, using multiples of the difference between two consecutive terms without a final adjustment and mixing variables. Along the study, this tendency was gradually inverted as most students understood the limitations of some of those strategies; (d) visualization proved to be a useful ability in different situations like making a drawing and counting its elements, to solve near generalization tasks, and "seeing" the structure of the pattern, finding an *explicit* strategy to solve far generalization tasks. So we think that it's important to provide tasks which encourage students to use and understand the potential of visual strategies and to relate number context with visual context to better understand the meaning of numbers and variables.

It is our strong belief that evidence about the ways children work with patterns may contribute to significant teaching decisions, about the ways to increase mathematical knowledge in our students and particularly of algebraic thinking.

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